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On the use of the notion of suitable weak solutions in CFD

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SUMMARY

The notion of suitable weak solutions for the three-dimensional incompressible Navier–Stokes equations together with some standard regularization techniques for constructing these solutions is reviewed. The novel result presented in this paper is that Faedo–Galerkin weak solutions to the Navier–Stokes equations are suitable provided they are constructed using finite-dimensional spaces having a discrete commutator property and satisfying a proper inf–sup condition. Low-order mixed finite element spaces appear to be acceptable for this purpose. Connections between the notion of suitable solutions and LES modeling are investigated. A proposal for a large eddy scale model based on the notion of suitable solutions is made and numerically illustrated. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

1.1. Position of the problem

This paper focuses on the notion of suitable weak solutions for the three-dimensional incompressible Navier–Stokes equations and discusses the relevance of this notion to computational fluid dynamics (CFD).

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Let Ω be a connected, open, bounded domain in $\mathbb{R}^3$ and consider the Navier–Stokes equations in Ω

$$\begin{aligned} \partial_t \mathsf{u} + \mathsf{u}\cdot\nabla\mathsf{u} + \nabla\mathsf{p} - R_e^{-1}\Delta\mathsf{u} = \mathsf{f} \quad \text{in } Q_T \\ \nabla\cdot\mathsf{u} = 0 \quad \text{in } Q_T \\ \mathsf{u}|_{t=0} = \mathsf{u}_0, \quad \mathsf{u}|_\Gamma = 0 \end{aligned} \tag{1}$$

where $Q_T = \Omega\times(0,T)$ is the space–time domain, Γ is the boundary of Ω, and R_e is the Reynolds number. Additional regularity requirements on f and u_0 will be added when needed.

The question we want to address in this paper is that of constructing approximate solutions to (1) using under-resolved meshes. Solving (1) on under-resolved meshes cannot be avoided when the Reynolds number is large, which is very often the case in engineering situations. At the present time, simulating time-dependent flows at Reynolds numbers greater than a few thousands is a daunting task. The reason for the very limited success of direct numerical simulation (DNS) is rooted in the heuristic Kolmogorov estimate $\mathcal{O}(R_e^{9/4})$ for the total number of degrees of freedom required to simulate flows at a given value of R_e. Considering the current pace of progress in computing power, this estimate undercuts the prospect of DNS of large-Reynolds number flows to some date possibly far in the future.

The current trend in CFD consists of computing the scales of the flow that can be represented on the available grid (i.e. the so-called large eddy scales (LES)) and modeling the rest of the scales that are smaller than the mesh size. The modeling in question has spawned scores of schools of thoughts, not necessarily converging. An extended variety of LES models is now available; see e.g. [1–3] for reviews. However, no satisfactory mathematical theory for LES has yet been proposed (for preliminary attempts of formalization see [3–5]). More surprisingly, no mathematical definition of LES has been stated either besides that in [6] and that in Hoffman and Johnson [7, 8].

In the wake of [6, 8], the objective of the present paper is to show that the notion of suitable weak solutions introduced by Scheffer [9] is a sound, firm, mathematical ground, which may be useful to LES modelers to built up their theories. In particular, we propose at the end of the present paper a model that aims at controlling the energy balance at the grid scale in a way that is consistent with the notion of suitable solutions, i.e. energetically consistent.

The paper is organized as follows. The definition of suitable weak solutions to the Navier–Stokes equations is recalled in Section 2. Some open questions regarding this notion are discussed. Standard techniques to construct suitable weak solutions are detailed in Section 3. All these techniques are shown to have striking similarities with popular LES methods. It is shown in Section 4 that the limits of Faedo–Galerkin approximations are suitable weak solutions to the Navier–Stokes equations provided the discrete spaces have a discrete commutator property and satisfy a proper inf–sup condition. This property holds for finite elements and wavelets, but is not satisfied by spectral methods. Hence, whether DNS spectral methods give suitable solutions at the limit is still an open question. The relevance of the notion of suitable weak solutions to under-resolved CFD is discussed in Section 5. It is shown therein that requiring discrete solutions to be suitable in a discrete sense amounts to requiring the energy of eddies of size similar to that of the mesh size to cascade down to subgrid scales and be eventually dissipated. A proposal for a LES model based on suitability is made at the end of the paper. The model consists of constructing a numerical viscosity measuring the default to suitability. The idea is illustrated numerically on the one-dimensional compressible Euler equations.

1.2. Notations and conventions

Spaces of $\mathbb{R}^3$-valued functions on Ω are denoted in bold font. No notational distinction is made between $\mathbb{R}$-valued and $\mathbb{R}^3$-valued functions. The Euclidean norm in $\mathbb{R}^3$ is denoted by $|\cdot|$, the maximum norm in $\mathbb{R}^3$ is denoted by $|\cdot|_\infty$. In the following c is a generic constant that may depend on the data f, u_0, R_e, Ω, T. The value of c may vary at each occurrence. Whenever E is a normed space, $\|\cdot\|_E$ denotes a norm in E. The scalar product in $L^2(\Omega)$ is denoted with parentheses, i.e. $(v,w) := \int_\Omega v(x)w(x)\,\mathrm{d}x$; the same notation is used for the scalar product in $\mathbf{L}^2(\Omega)$.

To account for solenoidal vector fields we set [10]

$$\mathbf{H} = \{v \in \mathbf{L}^2(\Omega);\ \nabla\cdot v = 0;\ v\cdot n|_\Gamma = 0\} \tag{2}$$

$$\mathbf{V} = \{v \in \mathbf{H}^1(\Omega);\ \nabla\cdot v = 0;\ v|_\Gamma = 0\} \tag{3}$$

2. SUITABLE WEAK SOLUTIONS

2.1. The definition

It is known since Leray [11] and Hopf [12] that weak solutions to (1) exist, but the question of uniqueness of these solutions is still open. This is an outstanding thorn in the side of mathematicians. The major obstacle in the way is that the *a priori* energy estimates obtained so far do not preclude the occurrence of so-called vorticity bursts reaching scales smaller than the Kolmogorov scale. It is in general believed that the uniqueness question is intimately intermingled with what is known/observed as turbulence. In mathematical terms, the turbulence question is an elusive one. Since the bold definition of turbulence by Leray in the 1930s [11], calling 'solution turbulente' any weak solution of the Navier–Stokes equations, progress has been frustratingly slow.

Since we are not able (yet) to prove uniqueness, we have to admit the fact that weak solutions may not be unique. Then one may wonder whether it is possible to distinguish one of these to be more physically relevant than the others. This principle has been fruitful for the analysis of scalar nonlinear conservations laws. It is known since Kružkov [13] that, although nonlinear conservations laws have infinitely many weak solutions in general, there is only one weak solution that satisfies the entropy inequality(ies). Hence, adding to the conservation law one entropy inequality (or more) is enough to select the so-called physical weak solution. One may then wonder whether a similar idea could apply for the Navier–Stokes equations. A possibly important step in this direction has been made by Scheffer [9].

In a ground-breaking paper [9] Scheffer introduced the notion of suitable weak solutions for the Navier–Stokes equations. In a few words, this notion boils down to the following.

Definition 2.1
Let (u,p), $\mathsf{u} \in L^2((0,T);\mathbf{H}^1(\Omega)) \cap L^\infty((0,T);\mathbf{L}^2(\Omega))$, $\mathsf{p} \in \mathscr{D}'((0,T);L^2(\Omega))$, be a weak solution to the Navier–Stokes equation (1). The pair (u,p) is said to be suitable if the local energy balance

$$\partial_t(\tfrac{1}{2}\mathsf{u}^2) + \nabla\cdot((\tfrac{1}{2}\mathsf{u}^2 + \mathsf{p})\mathsf{u}) - R_e^{-1}\Delta(\tfrac{1}{2}\mathsf{u}^2) + R_e^{-1}(\nabla\mathsf{u})^2 - \mathsf{f}\cdot\mathsf{u} \leqslant 0 \tag{4}$$

is satisfied in the distributional sense, i.e. in $\mathscr{D}'(Q_T;\mathbb{R}^+)$.

It is remarkable that the above inequality has striking similarities with entropy conditions for conservation laws. Think of it as an entropy inequality where the kinetic energy would play the

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role of an entropy (recall that for the inviscid Burgers equation, the kinetic energy is the only entropy that needs to be taken of).

With this notion Scheffer has been able to derive a bound from above of some Hausdorff measure of the set of singularities of suitable weak solutions, the remarkable fact being that these bounds cannot (yet) be obtained without invoking suitability, i.e. it is not known whether every weak solution satisfies (4). The result of Scheffer has been improved by Caffarelli–Kohn–Nirenberg and is now referred to as the Caffarelli–Kohn–Nirenberg Theorem [14, 15] in the literature. In a nutshell, this result asserts that the one-dimensional Hausdorff measure of the set of singularities of a suitable weak solution is zero. In other words, if singularities exist, they must lie on a space–time set whose dimension is smaller than that of a space–time line. At the present time, this is the best partial regularity result available for the Navier–Stokes equations. For any practical purpose, this theorem asserts that suitable weak solutions are almost classical. Recall that if a classical solution to (1) exists, then it is unique (of course, proving the existence of a classical solution is as difficult as proving the uniqueness of a weak solution). The word 'almost' is important here; although suitable weak solutions are the most regular solutions we know that exist, they may still have singular points, i.e. not be classical.

2.2. Open questions

For the person who is not familiar with this field, the first question that comes to mind is whether suitable weak solutions exist at all. The answer to this question is always yes. More details on how suitable solutions can be constructed are given later in Section 3. In a few words it suffices to regularize the Navier–Stokes equations just a little to obtain suitable solutions. This can be done either by smoothing the nonlinear term or by adding some hyperviscosity. Another way, which has been discovered in 2006, consists of constructing Galerkin approximations to (1). Under some hypotheses on the discrete setting, it can be shown that Galerkin solutions converge to suitable weak solutions as the mesh size goes to zero.

Another more intriguing question is the following: Is the class of suitable weak solutions a proper subclass of weak solutions? This problem seems to have been open since Scheffer introduced the notion of suitable solutions. Proving that these two sets are different would imply that the Navier–Stokes equations are not well posed, i.e. weak solutions are not unique (for some set of data, there would be a weak solution that is not suitable and one that is suitable); see Figure 1.

A third question a CFD specialist is certainly entitled to raise is the following: What use can be made of the notion of suitable weak solution in CFD? I hope it will be clear at the end of

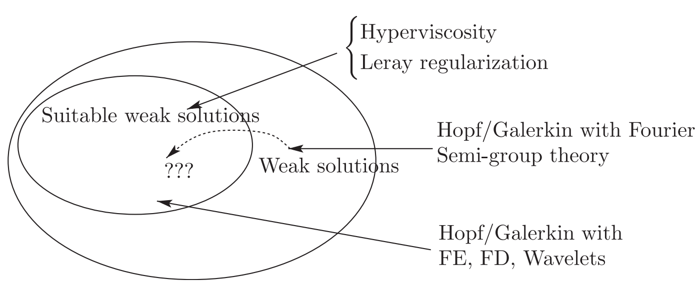

Figure 1. Suitable versus weak solutions.

this paper that suitability is exactly what a CFD specialist cares about the most. In a few words suitability is a guaranty that energy is dissipated at the mesh scale in a way that is physically consistent, i.e. the energy that goes below the mesh scale is dissipated and never returns.[§] See also Hoffman and Johnson [8, Chapter 19] for a related discussion on this topic.

Another question of interest is that concerning DNS. Since DNS is the highest court in the LES world, is it clear that as the mesh size goes to zero DNS solutions converge to a suitable solution? This is indeed true if the numerical method that is used satisfies some particular properties that will be detailed later in Section 4.2. Surprisingly, the Fourier method, which is the method of choice in turbulence computing, does not satisfy these properties, and to the best of my knowledge it is still an open question whether Fourier solutions are suitable at the limit.

3. CONSTRUCTION OF SUITABLE SOLUTIONS BY REGULARIZATION

The purpose of this section is to review some techniques that are known to produce suitable weak solutions. They all consist of regularizing the Navier–Stokes equations appropriately. All these technique have striking similarities with popular LES models. Why is it so? One can also reformulate the question otherwise: why is that most LES models have striking similarities with regularization methods that are known to produce suitable weak solutions? Would not it be that the LES world is looking for suitability without being aware of Definition 2.1?

3.1. Leray mollification

A simple construction yielding a suitable solution has indeed been proposed by Leray [11] before this very notion was introduced in the literature by Scheffer [9]. Leray proved the existence of weak solutions by using a, now very popular, mollification technique.

Assume that Ω is the three-dimensional torus $(0,2\pi)^3$. Denoting by $B(0,\varepsilon)\subset\mathbb{R}^3$ the ball of radius ε centered at 0, consider a sequence of nonnegative mollifying functions $(\phi_\varepsilon)_{\varepsilon>0}$ satisfying

$$\phi_\varepsilon\in\mathscr{C}_0^\infty(\mathbb{R}^3;\mathbb{R}_+),\quad \operatorname{supp}(\phi_\varepsilon)\subset B(0,\varepsilon),\quad \int_{\mathbb{R}^3}\phi_\varepsilon(\mathbf{x})\,\mathrm{d}\mathbf{x}=1 \tag{5}$$

Defining the convolution product $\phi_\varepsilon * v(\mathbf{x})=\int_{\mathbb{R}^3}v(\mathbf{y})\phi_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}$, Leray suggested to regularize the Navier–Stokes equations as follows:

$$\begin{aligned}\partial_t \mathsf{u}_\varepsilon+(\phi_\varepsilon * \mathsf{u}_\varepsilon)\cdot\nabla\mathsf{u}_\varepsilon+\nabla\mathsf{p}_\varepsilon-R_e^{-1}\Delta\mathsf{u}_\varepsilon&=\phi_\varepsilon * \mathsf{f}\\ \nabla\cdot\mathsf{u}_\varepsilon&=0\\ \mathsf{u}_\varepsilon \text{ is periodic}&\\ \mathsf{u}_\varepsilon|_{t=0}&=\phi_\varepsilon * \mathsf{u}_0\end{aligned} \tag{6}$$

The following holds (see [11, 16]):

[§]The reader who believes that it is legitimate to let energy come back from scales that cannot be represented by the mesh would save his time by stopping reading at this point. This statement does not deny backscatter, but simply says that if backscatter occurs and is important, then the mesh should rather be fine enough so that backscatter occurs above the finest mesh scale.

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Theorem 3.1
For all $\mathsf{u}_0 \in \mathbf{H}$, $\mathsf{f} \in \mathbf{H}$, and $\varepsilon > 0$, (6) has a *unique* $\mathscr{C}^\infty$ solution. The velocity is bounded uniformly in $L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$ and, up to subsequences, converges weakly in $L^2(0,T;\mathbf{V})$. The limit solution as $\varepsilon \to 0$ is a suitable weak solution of the Navier–Stokes equations.

The above mollification technique can be extended to account for the homogeneous Dirichlet boundary condition as done in [14] and the limit solution is suitable in this case as well.

Roughly speaking, the convolution process removes scales that are smaller than ε. Hence, by using $\phi_\varepsilon * \mathsf{u}_\varepsilon$ as the advection velocity, scales smaller than ε are not allowed to be nonlinearly active. This feature is a characteristic of most LES models for which ε would be the so-called LES.

3.2. NS-α and Leray-α models

Let ε be a small parameter that we henceforth refer to as the LES. Introduce the so-called Helmholtz filter $\overline{(\cdot)} : v \mapsto \overline{v}$ such that

$$\overline{v} := (I - \varepsilon^2 \Delta)^{-1} v \tag{7}$$

where either homogeneous Dirichlet boundary conditions or periodic boundary conditions are enforced depending on the setting considered. The so-called Navier–Stokes-alpha model introduced in Chen *et al.* [17] and Foias *et al.* [18, 19] consists of the following:

$$\begin{gathered} \partial_t \mathsf{u}_\varepsilon + \overline{\mathsf{u}_\varepsilon} \cdot \nabla \mathsf{u}_\varepsilon + (\nabla \overline{\mathsf{u}_\varepsilon})^{\mathrm{T}} \cdot \mathsf{u}_\varepsilon - R_e^{-1} \Delta \mathsf{u}_\varepsilon + \nabla \pi_\varepsilon = \mathsf{f} \\ \nabla \cdot \overline{\mathsf{u}_\varepsilon} = 0 \\ \mathsf{u}_\varepsilon|_\Gamma = 0, \quad \overline{\mathsf{u}_\varepsilon}|_\Gamma = 0 \quad \text{or} \quad \mathsf{u}_\varepsilon \text{ and } \overline{\mathsf{u}_\varepsilon} \text{ are periodic} \\ \mathsf{u}_\varepsilon|_{t=0} = \mathsf{u}_0 \end{gathered} \tag{8}$$

Once again, regularization yields uniqueness for each $\varepsilon > 0$ and suitability at the limit as stated in the following

Theorem 3.2
Assume $\mathsf{f} \in \mathbf{H}$, $\mathsf{u}_0 \in \mathbf{V}$. Problem (8) with the Helmholtz filter (7) has a unique solution u_ε in $\mathscr{C}^0([0,T);\mathbf{V})$ with $\partial_t \mathsf{u}_\varepsilon \in L^2(0,T;\mathbf{H})$. The solution $\overline{\mathsf{u}_\varepsilon}$ is uniformly bounded in $L^\infty(0,+\infty;\mathbf{H}) \cap L^2(0,+\infty;\mathbf{V})$ and, up to subsequences, converges weakly in $L^2_{\mathrm{loc}}(0,+\infty;\mathbf{V})$ to a weak Navier–Stokes solution as $\varepsilon \to 0$.

Here again, it is a simple matter to show that when periodic boundary conditions are enforced u_ε converges, up to subsequences, to a suitable solution.

A variant of the above regularization technique consists of replacing the term $(\nabla \overline{\mathsf{u}_\varepsilon})^{\mathrm{T}} \cdot \mathsf{u}_\varepsilon$ in (8) by $\nabla \frac{1}{2} \mathsf{u}_\varepsilon^2$. The resulting model then falls in the class of the Leray regularization in the sense that the momentum equation is the same as that in (6) but for the advection velocity $\phi_\varepsilon * \mathsf{u}_\varepsilon$, which is replaced by $\overline{\mathsf{u}_\varepsilon}$. This model has been analyzed in [20] and is called the Leray-α model. It also gives a suitable solution at the limit. It has been reported in [21] to be a good candidate for an LES model.

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3.3. Hyperviscosity

Lions [22, 23] proposed the following hyperviscosity perturbation of the Navier–Stokes equations:

$$\begin{aligned} \partial_t \mathsf{u}_\varepsilon + \mathsf{u}_\varepsilon \cdot \nabla \mathsf{u}_\varepsilon + \nabla \mathsf{p}_\varepsilon - R_e^{-1} \Delta \mathsf{u}_\varepsilon + \varepsilon^{2\alpha} (-\Delta)^\alpha \mathsf{u}_\varepsilon = \mathsf{f} \quad & \text{in } Q_T \\ \nabla \cdot \mathsf{u}_\varepsilon = 0 \quad & \text{in } Q_T \\ \mathsf{u}_\varepsilon|_\Gamma,\ \partial_n \mathsf{u}_\varepsilon|_\Gamma, \ldots, \partial_n^{\alpha-1} \mathsf{u}_\varepsilon|_\Gamma = 0 \quad & \text{or } \mathsf{u}_\varepsilon \text{ is periodic} \\ u|_{t=0} = u_0 \end{aligned} \tag{9}$$

where $\varepsilon > 0$ is the LES and $\alpha \in \mathbb{R}$. The appealing aspect of this perturbation is that it yields a well-posed problem in the classical sense when $\alpha \geqslant \frac{5}{4}$ in three space dimensions. More precisely, upon denoting the space dimension by $d \geqslant 2$, the following result (see [22–24]) holds.

Theorem 3.3
Assume $f \in L^2(0, T; \mathbf{L}^2(\Omega))$ and $u_0 \in \mathbf{H}^\alpha(\Omega) \cap \mathbf{H}_0^1(\Omega)$. Problem (8) has a unique solution u_ε in $L^\infty(0, T; \mathbf{H}^\alpha(\Omega) \cap \mathbf{H}_0^1(\Omega))$ for all times $T > 0$ if $\alpha > (d+2)/4$. Up to subsequences, u_ε converges to a weak solution u of (1), weakly in $L^2(0, T; \mathbf{H}_0^1(\Omega))$. Moreover, if periodic boundary conditions are enforced, the limit solution (u, p) is suitable.

It is remarkable that hyperviscosity models are frequently used in so-called LES simulations of oceanic and atmospheric flows [25–27].

3.4. Nonlinear viscosity models

Recalling that the Navier–Stokes equations are based on Newton's linear hypothesis, Ladyženskaja and Kaniel proposed to modify the incompressible Navier–Stokes equations to take into account possible large velocity gradients [28–30].

Ladyženskaja introduced a nonlinear viscous tensor $\mathbf{T}_{ij}(\nabla \mathsf{u})$, $1 \leqslant i, j \leqslant 3$ satisfying the following conditions:

$$\forall\ \xi \in \mathbb{R}^{3\times 3}, \quad |\mathbf{T}(\xi)| \leqslant c(1 + |\xi|^{2\mu})|\xi| \tag{10}$$

$$\forall\ \xi \in \mathbb{R}^{3\times 3}, \quad \mathbf{T}(\xi) : \xi \geqslant c|\xi|^2 (1 + c'|\xi|^{2\mu}) \tag{11}$$

$$\int_\Omega (\mathbf{T}(\nabla \xi) - T(\nabla \boldsymbol{\eta})) : (\nabla \xi - \nabla \boldsymbol{\eta}) \geqslant c \int_\Omega |\nabla \xi - \nabla \boldsymbol{\eta}|^2 \tag{12}$$

The three above conditions are satisfied if $\mathbf{T}(\xi) = \beta(|\xi|^2)\xi$, provided the viscosity function $\beta(\tau)$ is a positive monotonically increasing function of $\tau \geqslant 0$ and for large values of τ the following inequality holds: $c\tau^\mu \leqslant \beta(\tau) \leqslant c'\tau^\mu$, where $\mu \geqslant \frac{1}{4}$ and c, c' are some strictly positive constants.

After introducing the LES $\varepsilon > 0$, the modified Navier–Stokes equations take the form

$$\begin{aligned} \partial_t \mathsf{u}_\varepsilon + \mathsf{u}_\varepsilon \cdot \nabla \mathsf{u}_\varepsilon + \nabla \mathsf{p}_\varepsilon - \nabla \cdot (R_e^{-1} \nabla \mathsf{u}_\varepsilon + \varepsilon^{2\mu+1} \mathbf{T}(\nabla \mathsf{u}_\varepsilon)) = \mathsf{f} \\ \nabla \cdot \mathsf{u}_\varepsilon = 0 \\ \mathsf{u}_\varepsilon|_\Gamma = 0 \quad \text{or} \quad \mathsf{u}_\varepsilon \text{ is periodic} \\ \mathsf{u}_\varepsilon|_{t=0} = \mathsf{u}_0 \end{aligned} \tag{13}$$

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The main result from [29, 30] (see [28] for a similar result where monotonicity is also assumed) is the following theorem.

Theorem 3.4
Assume $\mathsf{u}_0 \in \mathbf{H}$ and $\mathsf{f} \in L^2(0,+\infty;\mathbf{L}^2(\Omega))$. Provided conditions (10)–(12) hold, then (13) has a unique weak solution, for all $\varepsilon>0$, in $L^{2+2\mu}(0,T;\mathbf{W}^{1,2+2\mu}(\Omega)\cap\mathbf{V})\cap C^0([0,T];\mathbf{H})$. Moreover, for periodic boundary conditions, $(\mathsf{u}_\varepsilon,\mathsf{p}_\varepsilon)$ converges, up to subsequences, to a suitable solution, of (1) as $\varepsilon\to 0$.

Possibly one of the most popular LES models is that proposed by Smagorinsky [31], which corresponds to setting $\mathbf{T}(\nabla\mathsf{u})=|\mathbf{D}|\mathbf{D}$. (i.e. $\beta(\tau)=\tau^\mu$ with $\mu=\frac{1}{2}$). Hence, in addition to other possible appealing features LES modelers may see in Smagorinsky-like LES models, the one of interest to us is that they guaranty well posedness for all $\varepsilon>0$ and give suitable solutions at the limit.

3.5. *Suitable versus weak*

Let us summarize what has been said so far. There are different ways of constructing weak solutions to (1).

The first strategy, developed by Leray [11], consists of introducing a small parameter ε and perturbing (1) so as to make sure that the perturbed problem is well posed (see all the examples in Section 3). Then by passing to the limit on ε one obtains a weak solution (actually, one obtains possibly many subsequences, each converging to a weak solution). Very often it turns out that the weak solutions thus reached are suitable.

A second strategy has been developed by Hopf [12] in 1951. Hopf showed that the Navier–Stokes equations have weak solutions with prescribed initial values in smooth bounded domains in the three-dimensional space, with zero velocity at the boundary. Hopfs proof does not make use of mollification but instead uses a Galerkin procedure to construct approximate solutions and reaches weak solutions by passing to the limit on the dimension of the approximation spaces, a technique that is now familiar to numerical analysts. Until recently it was not known whether the weak solutions reached by Hopf's technique are suitable. It turns out that this is indeed the case in some circumstances that are detailed in Section 4. Quite surprisingly, the case of Galerkin/Fourier approximations, which is very important to turbulence experts in CFD, is still an open problem.

A third different approach to the existence theory was taken by Fujita and Kato [32] in 1964 by making use of fractional powers of operators and the theory of semi-groups. It is not yet known whether the weak solutions reached by this theory are suitable.

To conclude (1) has always weak solutions and suitable weak solutions. Whether every weak solution is a suitable solution is an outstanding open question. The ideas discussed in this section are illustrated in Figure 1.

4. THE CASE OF FAEDO–GALERKIN APPROXIMATIONS

The purpose of this section is to clarify the situation of DNS. We want to answer the following question: Do approximate Navier–Stokes solutions computed with DNS methods converge to suitable weak solutions as the mesh size goes to zero? Using the language of Section 3.5, is it true that Hopf/Galerkin approximate solutions converge to weak suitable solutions?

4.1. The Galerkin setting

Let us introduce a family of discrete spaces $\mathbf{X}_h \subset \mathbf{H}_0^1(\Omega)$ for the velocity and a family of discrete spaces $M_h \subset H^1_{\int=0}(\Omega) := \{q \in H^1(\Omega);\ \int_\Omega q = 0\}$ for the pressure.

In addition to the usual interpolation (approximation) properties of $\mathbf{X}_h$ and M_h that we omit here for conciseness, we assume that the pair $(\mathbf{X}_h, M_h)$ is compatible in the following sense: There is $c>0$ independent of h such that

$$\forall q_h \in M_h, \qquad \sup_{0\neq v_h\in \mathbf{X}_h} \frac{(\nabla q_h, v_h)}{\|v_h\|_{\mathbf{L}^2}} \geqslant c\|\nabla q_h\|_{\mathbf{L}^2}$$

This condition is a strengthened version of the Ladyzhenskaya-Babsuka-Brezzi (LBB) condition. It can be shown to hold for a large variety of elements that are H^1-conforming on the pressure (Hood-Taylor, Mini, etc.).

Since the velocity field may not be solenoidal (recall that the mass conservation is enforced weakly only), we must slightly modify the nonlinear term to make it skew-symmetric. This can be done at least in two ways as follows:

$$b_h(u, v, v) = \begin{cases} (u\cdot\nabla u + \frac{1}{2}u\nabla\cdot u, v) & [33] \\ ((\nabla\times u)\times u + \frac{1}{2}\nabla(\mathscr{K}_h(u^2)), v) \end{cases}$$

where $\mathscr{K}_h : L^2(\Omega) \longrightarrow M_h$ is any linear L^2-stable approximation operator (for instance, the L^2-projection).

Using the above setting, the Hopf/Galerkin formulation of (1) is the following: Seek $u_h \in \mathscr{C}^1([0,T];\mathbf{X}_h)$ and $p_h \in \mathscr{C}^0([0,T]; M_h)$ such that for all $v_h \in \mathbf{X}_h$, all $q_h \in M_h$, and all $t \in [0,T]$

$$\begin{aligned} (\partial_t u_h, v) + b_h(u_h, u_h, v) - (p_h, \nabla\cdot v) + R_e^{-1}(\nabla u_h, \nabla v) &= \langle \mathsf{f}, v\rangle \\ (\nabla\cdot u_h, q) &= 0 \\ u_h|_{t=0} &= \mathscr{I}_h u_0 \end{aligned}$$

where $\mathscr{I}_h : L^2(\Omega) \longrightarrow V_h$ is any L^2-stable interpolation operator (it can be the L^2-projection, for instance). The pair (u_h, p_h) is henceforth referred to as a DNS solution. Note that at variance with the various regularization techniques presented in Section 3, no smoothing of the nonlinear term is done and no extra hyperviscosity is added.

4.2. The discrete commutator property

Before attacking the question of the convergence of the DNS approximation, we now make a fundamental assumption on the discrete setting. We assume that the discrete framework satisfies the following property that we henceforth refer to as the discrete commutator property (see Bertoluzza [34–36], [37, Appendix B], or [38, Chapter I.7]):

Definition 4.1
The space $\mathbf{X}_h$ (resp. M_h) is said to have the discrete commutator property if there is an operator $P_h \in \mathscr{L}(\mathbf{H}_0^1(\Omega); \mathbf{X}_h)$ (resp. $Q_h \in \mathscr{L}(H^1(\Omega); M_h)$) such that for all ϕ in $W_0^{2,\infty}(\Omega)$ (resp. all ϕ in $W_0^{2,\infty}(\Omega)$) and all $v_h \in \mathbf{X}_h$ (resp. all $q_h \in M_h$)

$$\begin{aligned} \|\phi v_h - P_h(\phi v_h)\|_{\mathbf{H}^l} &\leqslant ch^{1+m-l}\|v_h\|_{\mathbf{H}^m}\|\phi\|_{W^{m+1,\infty}}, \quad 0\leqslant l\leqslant m\leqslant 1 \\ \|\phi q_h - Q_h(\phi q_h)\|_{H^l} &\leqslant ch^{1+m-l}\|q_h\|_{H^m}\|\phi\|_{W^{m+1,\infty}} \end{aligned}$$

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Remark 4.1
When P_h (resp. Q_h) is a projector, the above definition is an estimate of the operator norm of the commutator $[\Phi, P_h] := \Phi \circ P_h - P_h \circ \Phi$ where $\Phi \circ v := \phi v$. This property is also called 'super-approximation' in the finite element literature [8, 35, 36].

Remark 4.2
The discrete commutator property is known to hold in discrete spaces where there exist projectors that have local approximation properties, see Bertoluzza [34]. It is known to hold for finite elements and wavelets. The key property is localization. To understand how the discrete commutator property can be proved let us assume that P_h is a linear projector and let $x \in \Omega$. For every y in a ball of radius h centered at x, we formally have $P_h(\phi v_h)(y) \approx P_h((\phi(x) + \mathcal{O}(h))v_h)(y) \approx (\phi(x) + \mathcal{O}(h))P_h(v_h)(y) = (\phi(x) + \mathcal{O}(h))v_h(y) + \mathcal{O}(h)$, that is to say $P_h(\phi v_h)(y) - (\phi v_h)(y) \approx \mathcal{O}(h) v_h(y)$, where $\mathcal{O}(h)$ depends on the gradient of ϕ.

We are now in a position to state the main result of this section.

Theorem 4.1
Under the above hypotheses, if $\mathbf{X}_h$ and M_h have the discrete commutator property, the pair (u_h, p_h) convergences, up to subsequences, to a suitable solution to (1).

The proof to this result is quite technical and can be found in [39]. The same result holds with periodic boundary conditions, see [37]. The technicalities in [39] reside in the handling of the Dirichlet conditions on the velocity. The discrete commutator property is the key argument in both [37, 39].

Theorem 4.1 asserts that, provided the discrete commutator property is satisfied by the underlying discrete setting, DNS solutions converge to suitable weak solutions. In other words, DNS solutions dissipate energy correctly at very fine scales. This result is not likely to surprise CFD specialists. However, there is still some surprise in store when it comes to understand what happens when one uses Fourier expansions to construct the DNS solution.

4.3. What happens with Fourier approximations?

Since most turbulence models are tested against DNS simulations using Fourier-based methods, we now need to focus our attention on the Fourier technique. We assume in this section that Ω is the three-dimensional torus.

Let N be a positive integer. The velocity and the pressure fields are approximated using trigonometric polynomials of partial degree less than or equal to N:

$$\mathbb{P}_N = \left\{ p(\mathbf{x}) = \sum_{|\mathbf{k}|_\infty \leqslant N} c_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, c_{\mathbf{k}} = \overline{c}_{-\mathbf{k}} \right\}$$

Since the mean value of the velocity and that of the pressure are irrelevant in the torus, we introduce $\dot{\mathbb{P}}_N$ the subspace of $\mathbb{P}_N$ composed of the trigonometric polynomials of zero mean value. Upon introducing the notation $h = 1/N$, we define

$$\mathbf{X}_h = \dot{\mathbf{P}}_N \quad \text{and} \quad M_h = \dot{\mathbb{P}}_N \tag{14}$$

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The stunning result of this section is summarized in the following:

Proposition 4.1
The Fourier setting does not have the discrete commutator property.

Proof
We construct a counter example. Consider now the very smooth function $\phi(x)=e^{i(x_1+x_2+x_3)}$ and set $v_h=(1,-\frac{1}{2},-\frac{1}{2})e^{iN(x_1+x_2+x_3)}$ (not that it matters, but note that $\nabla\cdot v_h=0$). Let $P_h:\mathbf{L}^2(\Omega)\longrightarrow\mathbf{X}_h$ be the $\mathbf{L}^2$-projection. Then $\phi P_h(v_h)=\phi v_h$ since $v_h\in\mathbf{X}_h$. Now, observing that ϕv_h is a trigonometric polynomial of degree $N+1$, we obtain $P_h(\phi v_h)=0$. As a result $[\phi, P_h](v_h)=\phi v_h$ and clearly $\|[\phi, P_h](v_h)\|_{\mathbf{L}^2}\geqslant c\|v_h\|_{\mathbf{L}^2}$, where $c>0$ depends only on ϕ. This means that the norm of the commutator operator $[\phi, P_h]$ in $\mathscr{L}(\mathbf{L}^2(\Omega);\mathbf{L}^2(\Omega))$ is bounded from below by c, i.e. it does not go to zero. This result holds for any other approximation operator, since the L^2-projection is the best approximation in $\mathbf{L}^2(\Omega)$. □

The consequence of this negative result is that the hypotheses of Theorem 4.1 do not hold, meaning that it not yet known whether Fourier approximations converge to suitable weak solutions. I think this question should attract the interest of mathematicians, since it might be a place to set a wedge that could separate the class of suitable weak solutions from that of those that are weak only.

One way to interpret the above results is the following: The Fourier technique is so accurate that it does not induce enough numerical diffusion to counteract the Gibbs–Wilbraham phenomenon. The key here is the lack of localization. On the other hand finite elements, wavelets, finite differences, etc. have enough built-in numerical dissipation to help the energy cascade to go in the right direction, meaning that the energy at extremely fine scales is always dissipated when using approximation methods having local interpolation properties.

5. UNDER-RESOLVED SIMULATIONS

The goal of this section is to explore some implications the notion of suitable solutions may have when it comes to approximate the Navier–Stokes equations on a finite grid. In other words, since $\lim_{h\to 0}$ is a mathematical dream, which is unachievable with the computing power currently available, can we anyway draw something useful from the existence of suitable solutions?

5.1. Practical interpretation of the notion of a suitable solution

At high Reynolds numbers CFD is always under-resolved, i.e. the Reynolds is always too large with respect to the mesh size at disposal. In other words, even if one uses finite elements, finite differences, wavelets, or any other setting admitting a discrete commutator property, the results of Theorem 4.1 are useless for practical purposes since the approximate solution thus calculated may be far from a (the?) suitable solution. The limit $h\to 0$ is an ideal situation from which practical CFD simulations are usually far. Then, one may ask oneself what is the use of the notion of suitable solutions? Is it a notion that we should care about in CFD?

To answer the above question, let us rephrase the definition of suitability. Let u, p be a weak solution of the Navier–Stokes equations in the Leray class (i.e. the velocity field satisfies the usual

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global energy estimates). Let us define the residual of the momentum equation:

$$R(x,t) := \partial_t \mathsf{u} - R_e^{-1}\Delta \mathsf{u} + \mathsf{u}\cdot\nabla \mathsf{u} + \nabla \mathsf{p} - \mathsf{f} \tag{15}$$

u, p being a weak solution means that the residual $R(x,t)$ is zero in the distribution sense. Is it then clear that the power of the residual, $R(x,t)\cdot\mathsf{u}$, is zero? Well, no, since it is not known whether u is smooth enough to be tested against $R(x,t)$. Think of the one-dimensional inviscid Burgers equation $\partial_t u + \frac{1}{2}\partial_x(u^2) = 0$, for instance. In the distribution sense $0 = R_B(x,t) := \partial_t u + \frac{1}{2}\partial_x(u^2)$, but the unique entropy equation is that which satisfies $R_B(x,t)u = \frac{1}{2}\partial_t u^2 + \frac{1}{3}\partial_x(u^3) \leqslant 0$. It is indeed true that $R_B(x,t)u = 0$ at points (x,t) where u is smooth, but in shocks $R_B(x,t)u$ is a negative Dirac measure. Similarly the definition of a suitable solution can be rephrased as follows: A suitable solution is one for which the power of the residual is negative, i.e.

$$\partial_t(\tfrac{1}{2}\mathsf{u}^2) + \nabla\cdot((\tfrac{1}{2}\mathsf{u}^2 + \mathsf{p})\mathsf{u}) - R_e^{-1}\Delta(\tfrac{1}{2}\mathsf{u}^2) + R_e^{-1}(\nabla \mathsf{u})^2 - \mathsf{f}\cdot\mathsf{u} \leqslant 0 \tag{16}$$

in the distribution sense in Q_T. The reader may verify by himself that indeed (16) is formally equivalent to $R(x,t)\cdot\mathsf{u} \leqslant 0$ (the term 'formally' meaning: in the optimistic hypothesis that u and p are smooth functions). In other words, if singularities occur, suitable solutions are such that these singularities dissipate energy.

5.2. What happens in under-resolved simulations?

Let us now focus our attention on numerical simulations and let us put ourselves in the under-resolved situation. Being under-resolved in a space–time region means that the numerical solution experiences large gradients that cannot be correctly represented by the mesh in the region in question. In other words, for all practical purposes, the numerical solution is singular at the considered mesh scale (i.e. behaves like a singular one on the available mesh). As time progresses the large unresolved gradients are likely to produce even larger gradients through nonlinear interactions, i.e. we have to deal with subgrid scales that can uncontrollably produce or dissipate energy locally. The question is no longer to determine whether the solution(s) to the Navier–Stokes equation is (are) classical or not (a debate that a pragmatic reader may think being of remote academic interest), it now just amounts to deciding what to do with a quasi-singular numerical solution.

Let us rephrase the situation in mathematical terms. Let (u_h, p_h) be the approximate velocity and the approximate pressure, the subscript h representing the typical mesh size. Let $D_h(x,t)$ be the numerical residual of the energy (entropy) equation:

$$D_h(x,t) := \partial_t(\tfrac{1}{2}u_h^2) + \nabla\cdot((\tfrac{1}{2}u_h^2 + p_h)u_h) - R_e^{-1}\Delta(\tfrac{1}{2}u_h^2) + R_e^{-1}(\nabla u_h)^2 - \mathsf{f}\cdot u_h \tag{17}$$

Being under-resolved in a neighborhood of (x_0, t_0) means that $D_h(x_0, t_0)$ is significantly larger than the consistency error of the method, i.e. $\|D_h(x_0,t_0)\| \gg 0$. If locally the power of the numerical singularity is negative, i.e. $D_h(x_0,t_0) \leqslant 0$, we do not have anything to fear since energy is cascading down and is eventually lost in the subgrid scales, a scenario in agreement with the Kolmogorov cascade. On the other hand if the numerical singularity produces energy, i.e. $D_h(x_0,t_0)\cdot u_h > 0$, all the bets are off since the situation is out of control and, by analogy with a shock that would produce energy, is unphysical.

In conclusion, ensuring that $D_h(x_0,t_0) \leqslant 0$ is a highly desirable feature. If it could be enforced everywhere in the domain, it would mean that the energy gently cascades down in the subgrid scales and is eventually dissipated. Rephrased in eddy terms, these conditions would guaranty that every eddy of size similar to the mesh size would eventually be dissipated.

It seems clear now that in under-resolved situations, one should wish the discrete solution to satisfy

$$D_h(x,t)\leqslant 0 \quad \forall (x,t)\in Q_T \tag{18}$$

i.e. the approximate solution should be suitable in the discrete sense.

5.3. Proposal for an LES model

Of course (18) cannot be enforced in addition to the discrete momentum equation and the discrete mass conservation. However, similarly to the entropy condition for nonlinear conservation laws, (18) must be incorporated in the algorithm that calculates the pair (u_h, p_h).

Possibilities are numerous. The first technique that comes to mind is to use (18) to construct an artificial viscosity:

$$\nu_h(u_h, p_h) := \min\left(R_e^{-1}, h\frac{(D_h(x,t))_+}{\|u_h\|\|\nabla u_h\|}\right) \tag{19}$$

where $t_+ := \frac{1}{2}(t+|t|)$ is the positive part. The momentum equation can then be modified by adding the term $-\nabla\cdot(\nu_h(u_h, p_h)\nabla u_h)$ in the left-hand side. Observe that $\nu_h(u_h, p_h)$ is a consistent viscosity; it is of order of the consistency error when the mesh is fine enough to resolve all the scales. The viscosity is zero when the power of the residual is negative, i.e. when the energy cascades down. The viscosity is active only in the under-resolved region if spurious energy is generated at the mesh scale, i.e. when energy seems to be coming up from subgrid scales.

5.4. Numerical illustrations

To support the idea that constructing a numerical viscosity based on the notion of entropy is a good one, we illustrate the concept on the compressible Euler equations in one space dimension. To make the challenge difficult, we use the Fourier method, which is the most unstable of all since it has no built-in numerical dissipation as discussed in Section 4.3 (see Guermond and Pasquetti [40] for details).

The Euler equations of gas dynamics in one space dimension are as follows:

$$\partial_t v(x,t)+\partial_x f(v(x,t))=0, \quad v(0)=v_0, \quad v=\begin{bmatrix}\rho\\ \rho u\\ E\end{bmatrix}, \quad f(v)=\begin{bmatrix}\rho u\\ \rho u^2+p\\ u(E+p)\end{bmatrix} \tag{20}$$

where ρ is the density of the gas, u is the velocity, $m := \rho u$ is the momentum, E is the total energy per unit volume, and $p=(\gamma-1)(E-\frac{1}{2}\rho u^2)$ is the pressure, $\gamma := 1.4$. The Euler system expresses the conservation of mass, momentum, and energy for a perfect gas. This system has infinitely many weak solutions, but only one of these satisfy the entropy inequality:

$$\partial_t S+\partial_x(uS)\geqslant 0 \tag{21}$$

where the physical entropy[¶] is defined by

$$S=\rho\log(p/\rho^\gamma) \tag{22}$$

[¶] We adopt the physical convention of the positive entropy.

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To construct a Fourier/Galerkin approximation to (20), we face the following question: where should the entropy-based numerical viscosity be introduced? To answer this question, we follow physics and look at the viscous case, that is at the Navier–Stokes system for which the momentum, energy are as follows:

$$\partial_t m + \partial_x(mu+p) = \partial_x(\mu\partial_x u) \tag{23}$$

$$\partial_t E + \partial_x((E+p)u) = \partial_x(\mu u\partial_x u) + \partial_x(\kappa\partial_x T) \tag{24}$$

where $T := p/\rho$ is the temperature, and μ and κ are the viscosity and conductivity, respectively. The Navier–Stokes entropy satisfies the following balance equation:

$$\partial_t S + \partial_x(uS - \kappa(\gamma-1)\partial_x \log(T)) = (\gamma-1)\left(\mu\frac{(\partial_x u)^2}{T} + \kappa(\partial_x \log T)^2\right) \tag{25}$$

This equation is the counterpart of (16) for the compressible Navier–Stokes equations.

Let $N \in \mathbb{N}$ be an integer. We then define an approximate Fourier/Galerkin solution to (20) by setting

$$v_N = \sum_{k=0}^{N-1} \hat{v}_k(t)\exp(\mathrm{i}kx), \quad \hat{v}_k(t) = \overline{\hat{v}}_{N-k} \tag{26}$$

$v_N(0) = P_N v_0$, and by solving

$$\partial_t v_N + \partial_x P_N(f(v_N) + f_{\text{visc}}(v_N)) = 0, \quad f_{\text{visc}}(v_N) = \begin{bmatrix} 0 \\ -\mu_N \partial_x v_N \\ -\mu_N u_N \partial_x u_N - \kappa_N \partial_x T_N \end{bmatrix} \tag{27}$$

where P_N is the L^2-projection onto the set of the trigonometric polynomials of degree at most N. We construct the numerical viscosity μ_N and the conductivity κ_N by following the same line of thought as in Section 5.2. We first define the entropy residual

$$\begin{aligned} R_N = {} & \partial_t S_N + \partial_x(u_N S_N - \kappa_N(\gamma-1)\partial_x \log(T_N)) \\ & -(\gamma-1)\left(\mu_N\frac{(\partial_x u_N)^2}{T_N} + \kappa_N(\partial_x \log T_N)^2\right) \end{aligned} \tag{28}$$

The quantity $|u_N| + (\gamma T_N)^{1/2}$ being the maximum wave speed, we construct a limiting viscosity as follows:

$$\mu_{\text{art}} = \alpha_r h \rho_N \max_x(|u_N(x)| + (\gamma T_N(x))^{1/2}) \quad \text{with } \alpha_r \in [\tfrac{1}{15}, \tfrac{1}{4}] \tag{29}$$

Then μ_N and κ_N are computed as follows:

$$\mu_N = \min(\mu_{\text{art}}, \alpha_m h L |R(v_N)|) \quad \text{with } \alpha_m \in [\tfrac{1}{4}, 2] \tag{30}$$

$$\kappa_N = \alpha_e \mu_N \quad \text{with } \alpha_e \in [0, \tfrac{1}{4}] \tag{31}$$

where L is the size of the computational domain and α_r, α_m, α_e are user-dependent parameters. Note that we can also use the negative part of the entropy residual, $(R_N)_-$, since definition (25) implies that S is an increasing quantity. Our experience is that using $(R_N)_-$ instead of $|R_N|$ slightly

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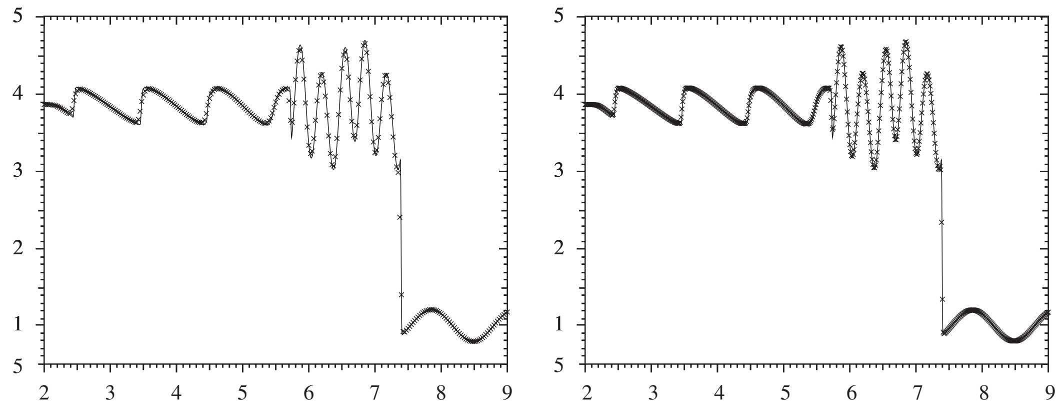

Figure 2. Shu–Osher shock tube, density at $t=1.8$, 400 (left) and 800 points (right).

sharpens shocks, although it is less robust (and using $(R_N)_+$ produces oscillations, as expected). However, again, using $|R_N|$ is slightly more robust since it can run very efficiently with very small values for α_r and α_m.

The time integration is done using the strongly stable explicit Runge–Kutta algorithm RK3 described in [41]. The nonlinear viscosity $v_N(u_N)$ is made explicit at each time step, say t^{k+1}, and evaluated at time t^k. To illustrate the performance of the above method, we now consider standard test problems.

We first consider the so-called Shu–Osher shock tube. The computational domain is $\Omega=[0,10]$, $L=10$, and the initial data are

$$\begin{cases} \rho = 3.857143, \quad v=2.629367, \quad p=10.333333 \quad \text{if } x<1 \\ \rho = 1+0.2\sin(5x), \quad v=0, \quad p=1 \quad \text{if } x>1 \end{cases} \tag{32}$$

The problem is made periodic by extending the domain to $[-10,10]$ and by slightly shifting the data and extending the shifted data. The entropy viscosity is constructed by using $|R_N|$ in the definition of μ_N. Using $(R_N)_-$ gives similar results. The computations are done with $\text{CFL}=0.1$, $\alpha_r=\frac{1}{8}$, $\alpha_r=1$, and $\alpha_e=\frac{1}{30}$. The graph of the density at $t=1.8$ is shown in the left panel of Figure 2 for two different resolutions, $N=200$ and 400 (400 and 800 points, respectively). This case is challenging since the fine features are very sensitive to any artificial viscosity. The reader familiar with this test case will note that it is remarkable that the solution with 400 points is very accurate.

The second test is the so-called Woodward–Collela shock wave. The computational domain is $\Omega=[0,1]$, $L=1$. The initial data are

$$\rho=1, \quad v=0, \quad \begin{cases} p=1000 & \text{if } x<0.1 \\ p=0.01 & \text{if } 0.1<x<0.9 \\ p=100 & \text{if } 0.9<x<1 \end{cases} \tag{33}$$

and the boundary conditions are $v|_{x=0}=0$ at $v|_{x=1}=0$ for all times. The Dirichlet boundary conditions are enforced by extending Ω to $[-2,2]$ and appropriately extending the data. The computations are done with $\text{CFL}=0.05$, $\alpha_r=\frac{1}{6}$, $\alpha_r=\frac{1}{2}$, and $\alpha_e=\frac{1}{20}$. The graph of the density at $t=0.038$ is shown in Figure 3 for various resolutions, 200, 400, 800, and 1600 points. We

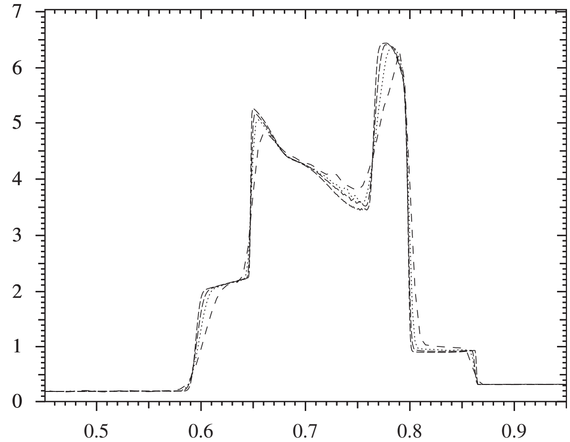

Figure 3. Woodward–Collela shock wave, density at $t=0.038$, using $200, 400, 800$, and 1600 points (divide by 2 to get N).

observe convergence and the reader who is familiar with this problem will recognize that the limit solution is the correct one and the method performs quite well when compared with other standard (nonadaptive) techniques available in the literature. In particular, the solution with 200 points is remarkably accurate considering that no adaptive mesh refinement has been done (recall that we are using the Fourier method).

We have applied the above strategy to two space dimension problems using conforming finite elements. We have performed a test on a two-dimensional transport equation. The velocity field is a constant rotation about the origin with angular speed equal to 2π. The initial data are the characteristic function of a disk of radius 0.25 initially centered at $(0.5, 0)$. The computation is done using conforming finite elements $\mathbb{P}_1$ and $\mathbb{P}_2$. The viscosity is defined by constructing the residual of the entropy equation using the square of the unknown as an entropy. We computed the error at time $t=1$ (one complete revolution) in the L^1-norm and the L^2-norm. We observed that the error measured in the L^1-norm behaves like $\frac{1}{2}(k+\frac{1}{2})/(k+1)$ in L^2, and $(k+\frac{1}{2})/(k+1)$ in L^1 where k is the degree of the finite elements ($k=1$ for conforming $\mathbb{P}_1$ elements and $k=2$ for conforming $\mathbb{P}_2$ elements). These estimates are compatible with the fact that the solution is in $H^{1/2-\varepsilon}$ and in $W^{1-\varepsilon,1}$ and they correspond to the L^p estimates that can be obtained by using a discontinuous Galerkin method (or any other standard stabilization technique), see [42, Theorem 4.2]. In other words, using a simple entropy-based viscosity stabilization is as efficient as DG and does not require additional shock capturing. We are currently working on the proof for this result. These findings will be reported elsewhere [40, 43].

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